Simple adaptive-feedback controller for identical chaos synchronization

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Based on the invariance principle of differential equations, a simple adaptive-feedback scheme is proposed to strictly synchronize almost all chaotic systems. Unlike the usual linear feedback, the variable feedback strength is automatically adapted to completely synchronize two almost arbitrary identical chaotic systems, so this scheme is analytical, and simple to implement in practice. Moreover, it is quite robust against the effect of noise. The famous Lorenz and Rössler hyperchaos systems are used as illustrative examples.

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Since it was shown in [1] that for some chaotic systems synchronization is possible, synchronization of (unidirectionally) coupled chaotic systems and its potential applications in engineering have been a field of great interest over a decade, see [2–4] and references cited therein. Due to the different applications, various specific synchronization schemes have been proposed in the literature, see [3] and references cited therein. However, just as was stated in [5], despite the large amount of effort, many key issues remain open. One of the central questions is, given two arbitrary identical chaotic systems, how can one design a physically available coupling scheme that is strictly guaranteed to produce stable identical synchronization motion (i.e., high-quality synchronization)? In most of the rigorous results based on the Lyapunov stability or the linear stability, the proposed scheme is very specific, but also the added controller is sometimes too big to be physically practical. One practical scheme is the linear feedback. However, in such a technique it is very difficult to find the suitable feedback constant, and thus numerical calculation has to be used, e.g., the calculation of the conditional Lyapunov exponents. Due to numerical calculation, such a scheme is not regular since it can be applied only to particular models. More unfortunately, it has been reported that the negativity of the conditional Lyapunov exponents is not a sufficient condition for complete chaotic synchronization, see [6]. Therefore, the synchronization based on these numerical schemes cannot be strict (i.e., high-qualitative), and is generally not robust against the effect of noise. Especially, in these schemes a very weak noise or a small parameter mismatch can trigger the desynchronization problem due to a sequence of bifurcations [7].

Actually, this open problem, although significant for complete chaos synchronization, is very difficult and cannot admit the optimization solution [3]. For example, in [5], rigorous criteria are presented to guarantee linearly stable synchronization motion, but the criteria are so complicated that specific numerical calculation is necessary for particular examples in practice. A similar problem was addressed in [8].

In this Brief Report, we give a novel answer to the above open problem. We prove rigorously by using the invariance principle of differential equations [9] that a simple feedback coupling with updated feedback strength, i.e., an adaptive-feedback scheme, can strictly synchronize two almost arbitrary identical chaotic systems.

Let a chaotic (drive) system be given as

$$\dot{x} = f(x), \tag{1}$$

where $x=(x_1,x_2,\ldots,x_n)\in \mathrm{R}^n$, $f(x)=(f_1(x), f_2(x),\ldots, f_n(x)):\mathrm{R}^n\to\mathrm{R}^n$ is a nonlinear vector function. And let $\Omega\subset\mathrm{R}^n$ be a chaotic bounded set of Eq. (1) which is globally attractive. For the vector function $f(x)$, we give a general assumption.

For any $x=(x_1,x_2,\ldots,x_n)$, $x_0=(x_1^0,x_2^0,\ldots,x_n^0)\in\Omega$, there exists a constant $l>0$ satisfying

$$|f_i(x)-f_i(x_0)|\leqslant l\max_j|x_j-x_j^0|,\ i=1,2,\ldots,n. \tag{2}$$

We call the above condition as the uniform Lipschitz condition, and l refers to the uniform Lipschitz constant. Note this condition is very loose, for example, the condition (2) holds as long as $\partial f_i/\partial x_j$ $(i,j=1,2,\ldots,n)$ are bounded. Therefore the class of systems in the form of Eqs. (1) and (2) include all well-known chaotic and hyperchaotic systems. Consider the variables of Eq. (1) as coupling signals, the receiver system with variables $y\in\mathrm{R}^n$ is given by the following equations:

$$\dot{y} = f(y) + \epsilon(y-x), \tag{3}$$

where the feedback coupling $\epsilon(y-x)=(\epsilon_1e_1,\epsilon_2e_2,\ldots,\epsilon_ne_n)$, $e_i=(y_i-x_i)$, $i=1,2,\ldots,n$ denotes the synchronization error of Eqs. (1) and (3). Instead of the usual linear feedback, the feedback strength $\epsilon=(\epsilon_1,\epsilon_2,\ldots,\epsilon_n)$ here will be duly adapted according to the following update law:

$$\dot{\epsilon}_i = -\gamma_i e_i^2, i=1,2,\ldots,n, \tag{4}$$

where $\gamma_i>0, i=1,2,\ldots,n$, are arbitrary constants. For the system of $2n$ equations (which is formally called the augment system for convenience below), consisting of the error equation between Eqs. (1) and (3) and Eq. (4), we introduce the following non-negative function:

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$$V=\frac{1}{2}\sum_{i=1}^{n} e_i^2+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(\epsilon_i+L)^2, \tag{5}$$

where L is a constant bigger than nl, i.e., $L > nl$. By differentiating the function V along the trajectories of the augment system, we obtain

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{n} e_i(\dot{y}_i-\dot{x}_i)+\sum_{i=1}^{n}\frac{1}{\gamma_i}(\epsilon_i+L)\dot{\epsilon}\\ &= \sum_{i=1}^{n} e_i[f_i(y)-f_i(x)+\epsilon_i e_i]-\sum_{i=1}^{n}(\epsilon_i+L)e_i^2\\ &\leqslant (nl-L)\sum_{i=1}^{n} e_i^2\leqslant 0.\end{aligned} \tag{6}$$

where we have assumed $x,y\in\Omega$ (without loss of the generality as Ω is globally attractive), and used the uniform Lipschitz condition (2). It is obvious that $\dot{V}=0$ if and only if $e_i=0, i=1,2,\ldots,n$, namely the set $E=\{(e,\epsilon)\in \mathrm{R}^{2n}: e=0,\epsilon=\epsilon_0\in \mathrm{R}^n\}$ is the largest invariant set contained in $\dot{V}=0$ for the augment system. Then according to the well-known invariance principle of differential equations [9], starting with arbitrary initial values of the augment system, the orbit converges asymptotically to the set E, i.e., $y\to x$ and $\epsilon\to\epsilon_0$ as $t\to\infty$.

Obviously, such identical synchronization motion is strict (i.e., high-qualitative), global (as long as the chaotic attractor is globally attractive), and nonlinearly stable. In particular, the nonlinear global stability implies that such chaos synchronization is quite robust against the effect of noise, namely under the case of presenting a small noise, the synchronization error eventually approaches zero and ultimately fluctuates around zero wherever the initial values start. In addition, we note that in order to reach synchronization, the variable feedback strength ϵ will be automatically adapted to a suitable strength ϵ_0 depending on the initial values, which is significantly different from the usual linear feedback. As is well known, in the usual linear feedback scheme a fixed strength is used wherever the initial values start, thus the strength must be maximal, which means a kind of waste in practice. Furthermore, the present control scheme does not require us to determine numerically any additive parameters, and is simple to implement in practice since the technique is similar to the well-known self-adaptive controller in the control theory. Note theoretically the converged feedback strength may be too big to be practical (although the added control may be small because the feedback error e_i may be small), but the flexibility of feedback strength in the present scheme can overcome this limitation once such a case arises. For example, suppose that the feedback strength is restricted not to exceed a critical value, say k. In the present control procedure, once the variable strength ϵ exceeds k at time $t=t_0$, we may choose the values of variables at this time as initial values and repeat the same control by resetting the initial strength $\epsilon(0)=0$. Namely, one may achieve synchronization within the restricted feedback strength due to the global stability of the present scheme, which is slightly similar to idea of OGY control [10]. This excellence is absent from the usual feedback scheme. In addition, in the present scheme the small converged strength may be obtained by decreasing suitably the value of parameter γ, which governs the rate of increasing the feedback strength. The question on how the parameter γ affects the convergence rate (i.e., transient time) and the final coupling strength remains to be further investigated.

Note that in the proposed scheme, the coupling of all the variables may be redundant to achieve synchronization, because we find from my proof that it is not necessary, e.g., one may set $\epsilon_i\equiv 0$ (i.e., canceling the corresponding coupling) if $|e_i|\leqslant|e_j|$. Actually, this case exists in general due to the nonhyperbolicity of a chaotic attractor, namely near a nonhyperbolic point the divergence rate of trajectories (the contraction and stretching of phase space along trajectories, respectively) in some directions vanishes (or is very small); see the following examples. Although we cannot give rigorous criteria to determine which variables can be used as the coupling signals for the general examples, in a concrete model one may determine it by the numerical technique, e.g., the calculation of the Lyapunov exponent respective to each direction. Of course for the low-dimensional systems the optimal coupling variables may be found by directly testing again and again. Especially, if based on the calculation of the conditional Lyapunov exponents a chaotic system can be synchronized by linearly coupling a variable, then the coupling of this variable in the present scheme can surely achieve synchronization; see the first example below.

Next we will give two illustrative examples. Consider the Lorentz system,

$$\begin{aligned}\dot{x}_1&=\beta(x_2-x_1),\\ \dot{x}_2&=\alpha x_1-x_1x_3-x_2,\\ \dot{x}_3&=x_1x_2-bx_3.\end{aligned} \tag{7}$$

The corresponding receiver system is

$$\begin{aligned}\dot{y}_1&=\beta(y_2-y_1)+\epsilon_1e_1,\\ \dot{y}_2&=\alpha y_1-y_1y_3-y_2+\epsilon_2e_2,\\ \dot{y}_3&=y_1y_2-by_3+\epsilon_3e_3\end{aligned} \tag{8}$$

with the update law (4). Now let $\beta=10,\alpha=28,b=\frac{8}{3}$. It has been well known that a suitable linear feedback in the second component may synchronize the Lorentz system, so we let $\epsilon_1=\epsilon_3\equiv 0$ (i.e., the time series of only the variable x_2 are selected as the driven signal), and set $\gamma_2=0.1$. The corresponding numerical results are shown in Figs. 1 and 2, where the initial feedback strength is set as zero. Figure 1 shows the temporal evolution of synchronization error between Eqs. (7) and (8) and the variable strength ϵ_2. Figure 2 shows that when an additive uniformly distributed random noise in the range $[-1,1]$ (i.e., a noise of the strength 1) is present in the signal output x_2 of Eq. (7), the synchronization error eventually approaches zero and ultimately fluctuates slightly around

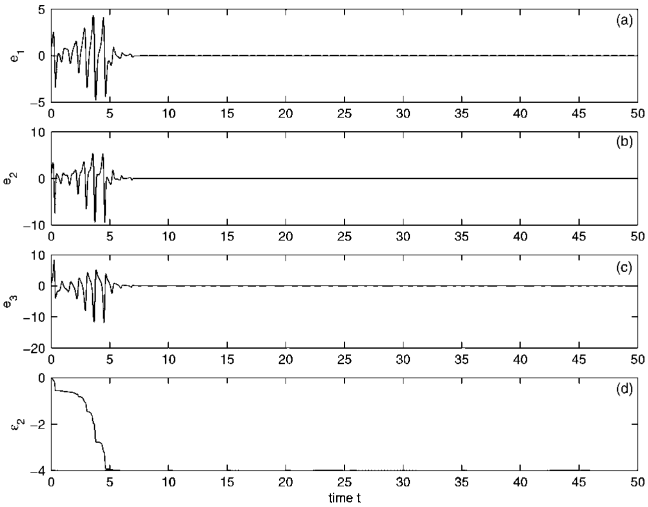

FIG. 1. The chaos synchronization between Eq. (7) and (8) is achieved by only the signal x_2, where (a)–(c) show temporal evolution of the synchronization error and (d) shows the evolution of the corresponding feedback strength ϵ_2. Here the initial values of (x,y,ϵ) are set as (2,3,7,3,4,8,0).

zero. Meanwhile, the variable feedback strength is affected slightly and cannot stabilize.

As the second example, we consider the Rössler hyperchaos system,

$$\dot{x}_1 = -x_2 - x_3, \quad \dot{x}_2 = x_1 + 0.25x_2 + x_4,$$

$$\dot{x}_3 = 3 + x_1 x_3, \quad \dot{x}_4 = -0.5x_3 + 0.05x_4. \tag{9}$$

The receiver system is

$$\dot{y}_1 = -y_2 - y_3 + \epsilon_1 e_1,$$

$$\dot{y}_2 = y_1 + 0.25y_2 + y_4 + \epsilon_2 e_2,$$

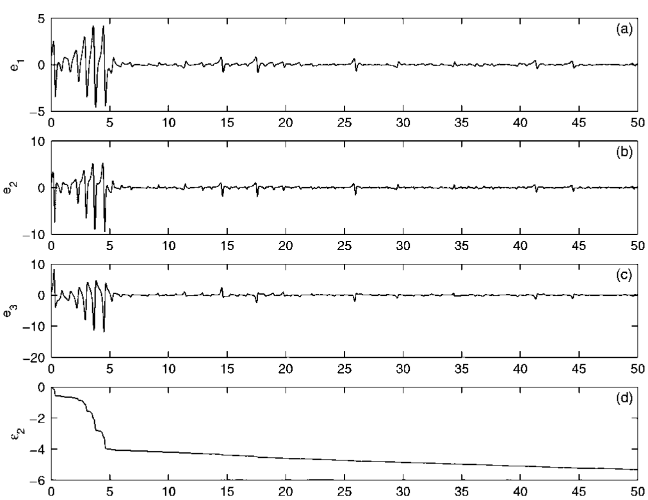

FIG. 2. (a)–(d) show the effect on the synchronization error and the feedback strength in Fig. 1 when a noise of the strength 1 is present in the signal output x_2 of Eq. (7).

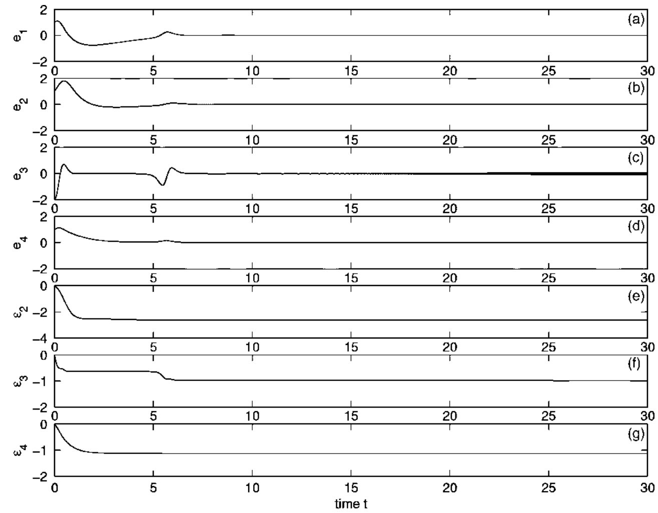

FIG. 3. (a)–(g) show the hyperchaotic synchronization between systems (9) and (10), and temporal evolution of the corresponding feedback strength, where only three variables of Eq. (9), x_i, $i=2,3,4$, are selected as the driven signals. Here the initial values of (x,y,ϵ) are set as (2,3,7,10,3,4,5,11,0,0,0).

$$\dot{y}_3 = 3 + y_1 y_3 + \epsilon_3 e_3,$$

$$\dot{y}_4 = -0.5y_3 + 0.05y_4 + \epsilon_4 e_4 \tag{10}$$

with the update law (4). Let $\epsilon_1 \equiv 0$ (we speculate that this is not optimal), $\gamma_i = 1, i = 2,3,4$, and initial feedback strength be (0,0,0). Numerical results for the two different cases are shown in Figs. 3 and 4, respectively. Figure 3 shows the hyperchaotic synchronization and Fig. 4 the slight effect of a noise with the strength 0.1 which is simultaneously added to the signals x_2, x_3, and x_4.

The above numerical examples show that chaotic or hyperchaotic synchronization can be quickly achieved by the present controller (i.e., the transient time to synchronization

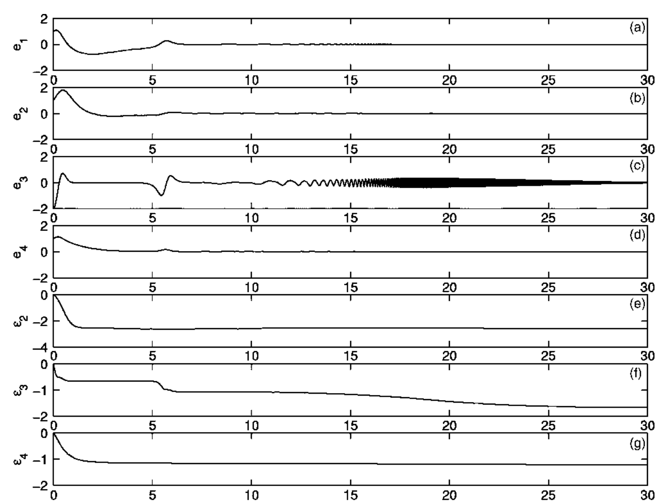

FIG. 4. (a)–(g) show the effect on the synchronization error and the feedback strength in Fig. 3 when a noise of the strength 0.1 is simultaneously added to the signal outputs x_2, x_3, and x_4 of Eq. (9).

is very short). In addition, we find from these examples that such synchronization is robust against the effect of noise, namely if the expected synchronization means that the synchronization error is eventually smaller than a threshold value (does not tend to zero), then the present scheme is physically feasible in the noisy case. Moreover, by comparing the converged feedback strength and the corresponding feedback signals, we find that the coupling is indeed small in the examples. In addition, by testing the other chaotic systems including the Rössler system, Chua's circuit, and Sprott's collection of the simplest chaotic flows, we find that the coupling of only one variable is sufficient to achieve identical synchronization of a three-dimensional system.

In conclusion, we have given a novel answer to an open problem in the field of identical chaos synchronization. In comparison with previous methods, the proposed scheme supplies a simple, analytical, and (systematic) uniform controller to synchronize strictly two arbitrary identical chaotic systems satisfying a very loose condition. The technique is simple to implement in practice, and quite robust against the effect of noise. The control idea may also be generalized to the case of the discrete chaotic systems. We also believe that such a simple synchronization controller will be very beneficial for the applications of chaos synchronization. Especially, the similar control scheme has been successfully used to stabilize the chaotic neuron model [11], and hence the proposed adaptive-feedback synchronization controller can be used to explore more reasonably the interesting dynamics found in neurobiological systems, i.e., the onset of regular bursts in a group of irregularly bursting neurons with different individual properties [12].

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